UNIVERSITÄT TRIER

GGWeek 2024

19th European Research Week on Geometric Graphs

August 11 - 16, 2024, Trier, Germany

ALGO

GGWeek 2024

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Program

	Monday	Tuesday	Wednesday	Thursday	Friday
7:45–9:00		breakfast	breakfast	breakfast	breakfast
9:00–10:30		group work	group work	group work	group work
10:30–11:00		coffee	coffee	coffee	coffee
11:00–12:30		group work	progress report	group work	progress report
12:30–14:00	lunch	lunch	lunch	lunch	lunch
14:00–15:30	open problem session	group work	group work	group work	wrap-up & farewell
15:30–16:00	coffee	coffee	coffee	coffee	
16:00–17:00	group work	group work	excursion	business meeting	
17:00–18:00				group work	
19:00–22:00			joint dinner (Bistro Walderdorffs)		

Participants

Oswin Aichholzer (TU Graz)
Todor Antić (Charles Uni Prague)
Patricia Bachmann (Uni Passau)
Martin Balko (Charles Uni Prague)
Helena Bergold (TU München)
Anna Brötzner (Malmö Uni)
Thomas Depian (TU Wien)
Simon Fink (TU Wien)
Henry Förster (Uni Tübingen)
Miriam Goetze (KIT)
Carolina Haase (Uni Trier)
Philipp Kindermann (Uni Trier)
Boris Klemz (Uni Würzburg)
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Marc van Kreveld (Uni Utrecht)
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Matthias Pfretzschner (Uni Passau)
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André Schulz (FernUni Hagen)
Diana Sieper (Uni Würzburg)
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Soeren Terziadis (TU Eindhoven)
Josef Tkadlec (Charles Uni Prague)
Alexandra Weinberger (FernUni Hagen)

Problem 1 Forbidden configurations in incidence graphs

suggested by Martin Balko

Let P be a set of n points and $\mathcal{L}$ a set of m lines in the plane. We define $I(P, \mathcal{L})$ to be the set of incidences between points from P and lines from $\mathcal{L}$. That is, $I(P, \mathcal{L})$ is the set of ordered pairs (p, L) such that $p \in P$, $L \in \mathcal{L}$, and $p \in L$. We write $G(P, \mathcal{L})$ to denote the incidence graph for P and $\mathcal{L}$. This (oriented) graph has vertex set $P \cup \mathcal{L}$ and edge set $I(P, \mathcal{L})$.

The celebrated Szemerédi–Trotter theorem [5] states that every set P of n points and every set $\mathcal{L}$ of m lines in the plane satisfies

$$|I(P, \mathcal{L})| \in O((mn)^{2/3} + m + n),$$

which is asymptotically tight. We focus on bounding the maximum number of point-line incidences in configurations $(P, \mathcal{L})$ that have some fixed forbidden induced subgraph in $G(P, \mathcal{L})$. This area was initiated by Solymosi [3].

Let P_1 and P_2 be two sets of points in the plane and $\mathcal{L}_1$ and $\mathcal{L}_2$ be two sets of lines in the plane. We say that $(P_1, \mathcal{L}_1)$ and $(P_2, \mathcal{L}_2)$ are isomorphic if the graphs $G(P_1, \mathcal{L}_1)$ and $G(P_2, \mathcal{L}_2)$ are isomorphic.

Question 1. *For some fixed set of points P_0 and for a set of lines $\mathcal{L}_0$ in the plane, what is the maximum number of incidences between n points and n lines in the plane containing no subconfiguration isomorphic to $(P_0, \mathcal{L}_0)$?*

Solymosi [3] proved the bound $o(n^{4/3})$ in the case that P_0 is a fixed set of k points in the plane in general position, that is, no three points from P_0 lie on a common line, and $\mathcal{L}_0$ is the set of all lines determined by points from P_0. This is so-called k-clique. The following problem was asked by Brass, Moser, and Pach [1, p. 291].

Question 2 ([1]). *For a k-clique $(P_0, \mathcal{L}_0)$, is there $\varepsilon > 0$ such that the maximum number of incidences between n points and n lines in the plane containing no k-clique is in $O(n^{4/3-\varepsilon})$?*

Suk and Tomon [4] also posed the following conjecture about k-fans and proved some lower bounds for the problem. Here, a k-fan consists of $k+1$ points and $k+1$ lines such that k points lie on a single line and the remaining k lines connect them to the $(k+1)$st point.

Conjecture 1 ([4])**.** *For every fixed $k \geq 3$, every set of n points and n lines in the plane that does not contain a k-fan determines at most $o(n^{4/3})$ incidences.*

The problem is open already for 3-fans, which is a case mentioned by Brass, Moser, and Pach [1, p. 291]. In general, one can consider various configurations, for example, Mirzaei and Suk [2] considered so-called natural $t \times t$ grids.

References

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Problem 2 Monochromatic convex 4-holes in bicolored point sets

suggested by Oswin Aichholzer

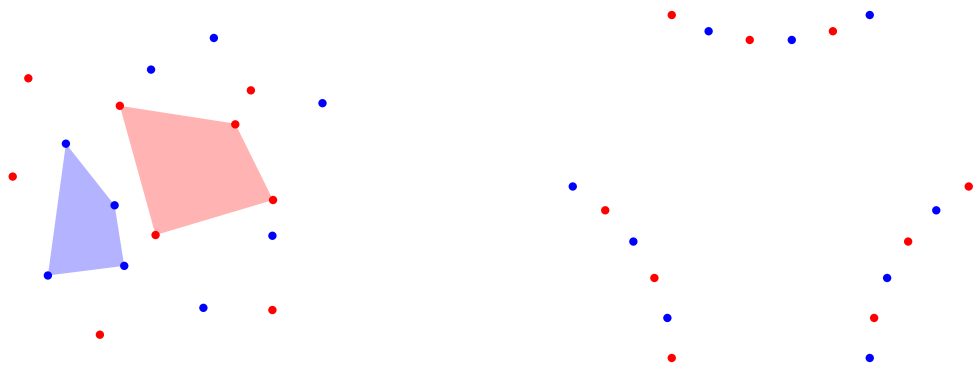

Figure 1: Left: Two monochromatic convex 4-holes. Right: A set of 18 points without monochromatic 4-hole.

Let S be a set of bicolored points in the plane in general position. A 4-hole h in S is a 4-gon spanned by points in S such that no other points of S are in the interior of h. The 4-hole h is called monochromatic if all 4 points of h belong to the same color class, and h is called convex, if the 4 points of H are in convex position.

Question 1. *Does there always exist a monochromatic convex 4-hole h in S?*

See Figure 1 for examples of monochromatic convex 4-holes (left), and a set of 18 points that does not contain a monochromatic 4-hole (neither convex nor non-convex). For the uncolored case it is known that for 5 or more points there always exists a convex 4-hole. In the bicolored case it hase been shown that for sufficiently large point sets there always exists a (not necessarily convex) monochromatic 4-hole [1]. The largest known bicolored point set that does not contain a convex monochromatic 4-hole has 36 points. If the points are colored with three or more colors, then it has been shown that there are sets not even spanning an empty monochromatic triangle. See [1] for more details, background, and relevant literature.

MONOCHROMATIC CONVEX 4-HOLES IN BICOLORED POINT SETS **Input:** A set of bicolored points in the plane in general position. **Question:** Does there always exist a monochromatic convex 4-hole?

References

[1] O. AICHHOLZER, T. HACKL, C. HUEMER, F. HURTADO, and B. VOGTENHUBER. Large bichromatic point sets admit empty monochromatic 4-gons. SIAM J. Discrete Math., 23(4):2147–2155, 2010. DOI: 10.1137/090767947.

Problem 3 Are geometric k-planar graphs geometric $(k+1)$-quasiplanar?

suggested by Todor Antić

We say that a drawing of a graph G is k-planar if every edge has at most k crossings with other edges of G. We say that a drawing of G is h-quasiplanar if it contains no set of h pairwise crossing edges. We say that G is (geometric) k-planar if it admits a (straight-line) drawing which is k-planar and that it is (geometric) h-quasiplanar if it admits a (straight-line) drawing which is h-quasiplanar. We want to study the relationship between these two classes of graphs. It is easy to notice that the following holds.

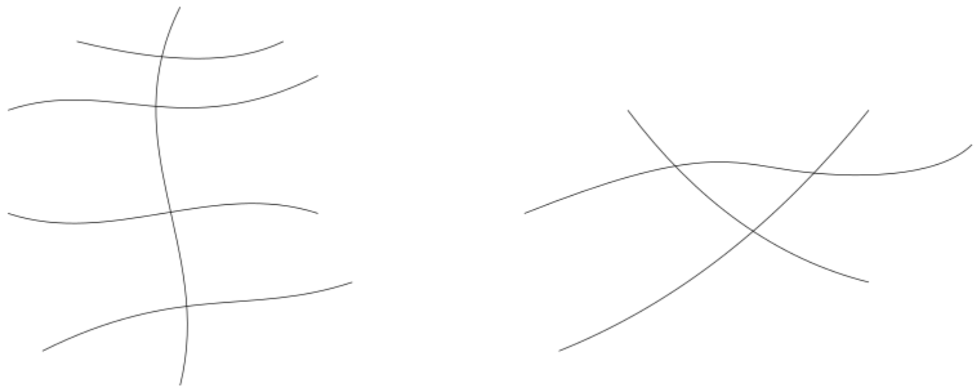

Figure 1: Forbidden configuration in 3-planar and 4-quasiplanar graphs

Exercise 1. *Each k-planar graph is $(k+2)$-quasiplanar.*

Still, if we want anything smaller than $k+2$, the problem becomes significantly harder. In the case where we insist that our drawings are simple, Angelini et al. [1] proved that every simple k-planar drawing can be redrawn to be simple $(k+1)$-quasiplanar. Their proof was quite involved and required multiple redrawing operations and a complicated analysis to prove their correctness. In a simpler case, where the drawings are straight-line and the vertices are in convex position, which we call convex-geometric drawings, it is possible to prove that every convex-geometric

k-planar drawing can be redrawn to be convex-geometric k-quasiplanar [2]. However, it is not clear how to change the methods used in either of these proofs to approach general geometric graphs. The proof for the case of simple drawings relies on a redrawing technique that does not change the position of the vertices but only redraws the edges and as such can not be applied to a more restrictive setting of geometric graphs. On the other hand, the proof for the convex-geometric case relies heavily on the combinatorial properties of convex-geometric graphs which do not translate to the more general case of geometric graphs. Therefore, it would be interesting to either solve or find a counterexample for the following question.

Question 1. *Is there a $k_0 \in \mathbb{N}$ such that if $k \geq k_0$ then every geometric k-planar graph is geometric $(k+1)$-quasiplanar.*

Of course, we can generalize this question and ask for a function $f : \mathbb{N} \to \mathbb{N}$ such that (for sufficiently large k), each geometric k-planar graph is $f(k)$-quasiplanar. It would be interesting to make this function as small as possible.

Some simple observations can be made that may help with solving Question 1.

Lemma 1. *Let X, Y be two sets of $k+1$ pairwise crossing edges in a k-planar drawing G then $X \cap Y = \emptyset$.*

Lemma 2. *Let G be a k-planar drawing of a graph and let X be a set of $k+1$ pairwise crossing edges of G. Then there is no edge of G which crosses an edge of X.*

In particular, the above lemmas tell us that we can assume that if G is a geometric k-planar drawing X is a set of $k+1$ pairwise crossing edges, we can assume that all of the vertices of G which are not in X lay in the outer face of the arrangement of line segments determined by X.

References

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Problem 4 (k-)Geometric Thickness of Complete Graphs

suggested by Henry Förster

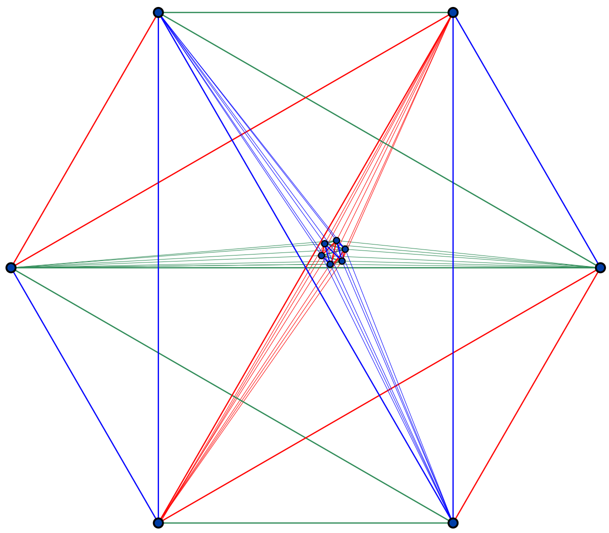

Figure 1: K_{12} has geometric thickness 3 [1].

Let $G = (V, E)$ be an abstract graph and let $\mathcal{E}$ a geometric embedding of G, that is, every vertex of V is assigned a coordinate in the plane and every edge of E is realized as a straight-line segment connecting both its endpoints. We say that $\mathcal{E}$ has *geometric thickness* t if the edges can be t-colored such that each pair of edges crossing in $\mathcal{E}$ is bicolored. Conversely, we say that G has *geometric thickness* t if it admits an embedding of geometric thickness t. We also write $\bar{\theta}(G)$ for the minimum geometric thickness of G.

Dillencourt, Epstein and Hirschberg [1] provided bounds on the geometric thickness of complete graphs; see Fig. 1 for an example embedding.

Theorem 1. $\lceil\frac{n}{5.646}\rceil \leq \bar{\theta}(K_n) \leq \lceil\frac{n}{4}\rceil$ *and* $\lceil\frac{ab}{2a+2b-4}\rceil \leq \bar{\theta}(K_{a,b}) \leq \lceil\frac{\min\{a,b\}}{2}\rceil$.

While the bound for bipartite complete graphs $K_{a,b}$ is tight for sufficiently large a (w.r.t. b), there is a significant gap for complete graphs K_n.

Question 1. *Can we close the gap between the upper and lower bound on* $\bar{\theta}(K_n)$*?*

It is worth remarking that Question 1 is open for 25 years by now. Thus, we may be tempted to first consider a potentially easier variant. To this end, we may consider the k-geometric setting where we seek an embedding of G in which every edge is a k-bend polyline. Let $\bar{\theta}_k(G)$ denote the k*-geometric thickness* of G, i.e., the minimum value t such that G admits a k-bend polyline embedding $\mathcal{E}$ such that its edges can be t-colored so that no two edges of the same color cross in $\mathcal{E}$.

Question 2. *What are upper and lower bounds for the* 1*-geometric thicknesses* $\bar{\theta}_1(K_n)$ *and* $\bar{\theta}_1(K_{a,b})$*? How about* $\bar{\theta}_k(K_n)$ *and* $\bar{\theta}_k(K_{a,b})$ *for* $k \geq 2$*?*

References

[1] M. B. Dillencourt, D. Eppstein, and D. S. Hirschberg. Geometric thickness of complete graphs. J. Graph Algorithms Appl., 4(3):5–17, 2000. doi: 10.7155/JGAA.00023.

Problem 5 Counting Crossing-Free Euler Tours

suggested by Günter Rote

We are interested in the number of "crossing-free" Euler tours of a plane graph.

Definition 1. *An* Euler tour *of a graph is a closed walk that uses every edge exactly once.*

For plane graphs, there are two notions of a "crossing-free" Euler tour.

Definition 2.

- *A* non-crossing *Euler tour is a tour that has no crossing: A crossing is formed by two pairs of consecutive edges* $(a, u), (u, b)$ *and* $(c, u), (u, d)$ *such that their radial order around* u *is* a, c, b, d *or* a, d, b, c. *(In a straight-line drawing, such a tour is a* weakly simple *polygon.)*
- *In an* A-trail *(sometimes called a* non-intersecting *Euler tour), every pair of consecutive edges* $(a, u), (u, b)$ *must be adjacent in the radial order around* u.

Exercise 1. *Show that every plane Eulerian graph (connected graph where every vertex has even degree) has a non-crossing Euler tour [5].*

On the other hand, for the stronger notion of A-trails, testing the existence is NP-hard [2], even for 3-connected graphs having only triangular and quadrilateral faces [1, Theorem 2].

We can ask the following types of questions:

A Extremal questions: How many non-crossing Euler tours/A-trails can a graph with n vertices and m edges have? At most? At least?

B Counting: How fast can we count non-crossing Euler tours?

The notions can be extended to multigraphs. In this case, we interpret them as if the multigraph were converted to a simple graph by subdividing every edge. Alternatively, we work directly with multigraphs, taking care the each edge in a bundle of parallel edges keeps its own identity.

Remark 1. *There are two models of counting Euler tours: We may either start at a vertex and proceed through all edges in some direction, or we may consider equivalence classes up to cyclic rotation and reversal of the tour. The difference is a fixed factor of* $2m$. *Thus, one may choose whichever model is more convenient and translate to the other model if necessary.*

Remark 2. *According to [3], Kotzig [4] showed that A-trails can be counted in polynomial time in 4-regular plane graphs.*

Remark 3. *The so-called* BEST Theorem[1] *of de Bruijn, Ehrenfest, Smith and Tutte from 1941/1951 gives a determinant-based formula for the number of Euler tours in directed multigraphs.*

References

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Problem 6 Reachability for Planar Unit-Length Linear Linkages in Simple Polygons

suggested by Thomas Depian, Simon Dominik Fink, Martin Nöllenburg

For this problem, we revisit the concept of linkages in $\mathbb{R}^2$ that has been extensively studied especially in the 80s and 90s [2, 5, 6, 9] and finds applications in robotic motion planning or protein folding. In the following, we use the nomenclature by Connelly and Demaine [1].

Definition 1 (Linkage)**.** *A linkage is a tuple (G, γ) where $G = (V, E)$ is a graph and $\gamma\colon E \to \mathbb{R}^+$ a function that assigns a positive length to each edge. If G is a path $(v_1, \ldots, v_{|V|})$, we call (G, γ) a linear linkage. A linkage (G, γ) is called unit-length if $\gamma(e) = \ell$ for all $e \in E$ and some $\ell \in \mathbb{R}^+$.*

Linear linkages are also known as rulers [5] and unit-length linear linkages as ℓ-rulers [9]. For a given linkage, we can look at drawings of G that satisfy the edge-lengths specified by γ.

Definition 2 (Configuration)**.** *A configuration Γ of a linkage (G, γ) is a straight-line drawing of G such that $\|\Gamma(e)\| = \gamma(e)$ holds for all $e \in E$.*

A configuration where no two edges cross is called planar. An ℓ-ruler that admits a planar configuration is also called a matchstick graph [7].

One problem that has often been considered is whether there exists a configuration, fixed at some point s, that reaches a given point $t \in \mathbb{R}^2$.

REACHABILITY
Input: A linear linkage (G, γ), a simple open polygon $P \subset \mathbb{R}^2$, and two points $s, t \in P$. **Question:** Does there exist a configuration Γ such that $\Gamma(v_1) = s$, $\Gamma(v_n) = t$, and the image of Γ is apart from s and t contained in P?

This problem is closely related to linkage reconfiguration (also known as folding), where the task is to transform a configuration Γ_1 into a config-

Linkage	Init. Conf. Γ_1?	Polygon P	Result	Reference
General	✔	None	PSPACE-hard	[4]
Linear	✔	4 segments	NP-hard	[5]
Linear	✔	Circle	P	[5]
Linear	✔	Simple	PSPACE-complete	[6]

Table 1: Overview of results on flavors of Reachability.

uration Γ_2. Often, each intermediate configuration should satisfy some property, e.g., be planar [1, 2]. Reachability can also be turned into a reconfiguration problem when we include an initial configuration Γ_1 (with $\Gamma_1(v_1) = s$) and the task is to reconfigure it into a configuration Γ that satisfies the stated properties. Indeed, in the literature, it is usually defined as a reconfiguration problem [1].

Different flavors of the Reachability problem have been extensively studied in the past, and Table 1 gives an overview. For more results related to linkages, consider the PhD theses of Pei [8] and Demaine [2] as well as the book (chapters) co-authored by Demaine [1, 3].

To obtain the hardness results listed in Table 1, the authors used non-unit-length linkages and a specific initial configuration. This gives rise to the following question.

Question 1. *What is the computational complexity of Reachability for unit-length linear linkages or without an initial configuration?*

Reachability does not make any restrictions on the polygon P or the points s and t. In particular, consider the case where (G, γ) is a unit-length linkage for some length ℓ and the edges of P also have length ℓ, i.e., P is a unit-length polygon. If s and t are also vertices of P, then the polygon itself witnesses a configuration that reaches t and is almost contained in P. This could make the problem easier and brings us to the next question.

Question 2. *What is the computational complexity of Reachability for* *unit-length linear linkages if P is a unit-length polygon and s and t are vertices of P?*

Finally, existing hardness results allow the configuration Γ to contain crossings. However, if linkages are used to model robot arms, then we require configurations to be planar. In this setting, the reconfiguration problem has been studied [1, 3]. However, to the best of our knowledge, the combination of planar configurations and bounding polygons has received little attention. It is therefore natural to ask which of the results from Table 1 also hold if we additionally require that no two edges cross in the configuration Γ.

Question 3. *What is the computational complexity of* Reachability *(for linear linkages) if we require the configuration Γ to be planar?*

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Problem 7 Crossing families

suggested by Alexandra Weinberger

Definition 1. *A crossing family in a geometric graph G is a set of edges in G that pairwise cross. The size of a crossing family, is the number of edges in that crossing family.*

We denote by $cf(G)$ the size of a largest crossing family in a geometric graph G.

Question 1. *What is the minimum $cf(G)$ over all complete bipartite geometric graphs G whose partition classes have size m and n?*

A graph is d-regular if each vertex has d neighbors.

Question 2. *What is the minimum $cf(G)$ over all geometric graphs G whose underlying abstract graph is a triangle-free d-regular graph on n vertices?*

For $d = 2$ the question is not yet exciting and to find out for which d the question gets interesting is part of the open problem.

The questions are inspired from research on crossing families in complete geometric graphs that has been done in the past decades. To the best of our knowledge, the currently best known bounds are the following (see the corresponding articles for references to previous bounds and work). Each complete geometric graph on n vertices contains a crossing family of size at least $n^{1-o(1)}$ [4]. There are complete geometric graphs on n vertices whose largest crossing family has size at most $8\lceil \frac{n}{41} \rceil$ [1].

A related result is that there exists a constant c (which cannot be taken larger than $\frac{2}{3}$) such that any set of k mutually crossing triangles in a geometric graph G contains a family of at least k^c mutually-crossing 2-paths (each of which is the result of deleting an edge from one of the triangles) [2].

But what if there are no triangles in the graph? Clearly, if the graph is planar, then there are drawings where the biggest crossing family has size one. We are interested in triangle-free graphs that have enough crossings to make looking for crossing families interesting, resulting in the above questions.

We remark that in the literature on crossing families in complete geometric graphs is mostly using the (equivalent) terminology of point sets in the plane.

We further remark that there is a very similar concept called intersection families (or in the topological world thrackles), where edges are pairwise either crossing or share a vertex. Intersection families have also been studied before considering complete graphs (and regular bipartite graphs have helped there [3]). We will not go father into the topic here, expect to mention that considering intersection families rather than crossing families in the here given questions would also lead to an interesting research problem.

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Problem 8 Edge-minimum Saturated Drawings of Geometrically Thick Graphs

suggested by Thomas Depian, Simon Dominik Fink, Soeren Terziadis

We suggest to combine two concepts from graph drawing, namely geometric thickness and saturated graph drawings.

Geometric Thickness. Tutte [7] introduced the minimum number of planar subgraphs whose union yields given a graph G as the thickness $\theta(G)$ of G. Dillencourt, Eppstein and Hirschberg [3] studied geometric thickness, a version of this concept which is more restricted in the sense that there must be a single straight-line drawing of the entire graph, which contains planar subdrawings.

Definition 1. *The geometric thickness $\bar{\theta}(G)$ of a graph G is the minimum number of colors s.t. there exists a straight-line drawing of G and an edge-coloring with $\bar{\theta}(G)$ colors that has no mono-chromatic crossings.*

Let Θ^k be the class of graphs G s.t. $\bar{\theta}(G) = k$. We call a drawing which is evidence of a graph having geometric thickness k a Θ^k-drawing. Deciding if the geometric thickness of a multigraph is larger than 30 has been proven $\exists\mathbb{R}$-complete [5] (result of GGWeek 2023). Solving the following open problem of said publication would prove the same bound for simple graphs, rather than multigraphs.

Question 1. *Given $t \in \mathbb{N}$, does there always exist a graph with geometric thickness t such that any Θ^t-drawing of G is connected in all t colors?*

This question intuitively requires the graphs to be somewhat dense in order to have "enough" edges in all layers. Related to this we introduce the next concept.

Edge Saturation. Saturation problems investigate under which condition it is impossible to add an edge to a graph in a graph class $\mathcal{C}$ s.t. the resulting graph is also in $\mathcal{C}$.

Definition 2. *A graph $G = (V, E)$ is saturated for a graph class $\mathcal{C}$ if there is no edge $e \notin E$ s.t. $(V, E \cup e) \in \mathcal{C}$. Saturated graphs are also called maximal.*

Within the set of saturated graphs we can look for minimal and maximal elements.

Definition 3. *Given $n \in \mathbb{N}$, a graph $G = (V, E)$ with $|V| = n$ is max-saturated (min-saturated) for a graph class $\mathcal{C}$ if it is saturated and there is no other $\mathcal{C}$-saturated graph with the same number of nodes but more (fewer) edges.*

Turán [6] and Erdős, Hajnal and Moon [4] investigated these graphs. For example planar-max-saturated graphs (which famously have at most $3n-6$ edges) coincide with planar-min-saturated graphs. But if edges are allowed to have one crossing, the number of edges in min- and max-saturated graphs can be different, i.e., $\frac{45}{17}n + O(1)$ vs. $4n - 8$ [1].

If a drawing can be fixed, then we can equally define min- and max-saturated drawings, i.e., drawings which cannot be extended with an additional edge s.t. the resulting drawing still conforms to some predefined drawing conventions. Chaplick et al. [2] have investigated this for k-planar graphs and present as an example a 4-planar drawing of a cycle of 8 vertices, which is saturated (while even the clique on 8 vertices allows for a 3-planar drawing).

Combining the concepts. Both k-planarity and geometric thickness are concepts that are related to the graph being "drawable" in a particular manner. Translating the concept of saturation from k-planarity to geometric thickness, we now aim to combine Θ^k-drawings with saturated drawings and state the following question.

Question 2. *What is the smallest number of edges in a graph that admits a saturated Θ^k-drawing?*

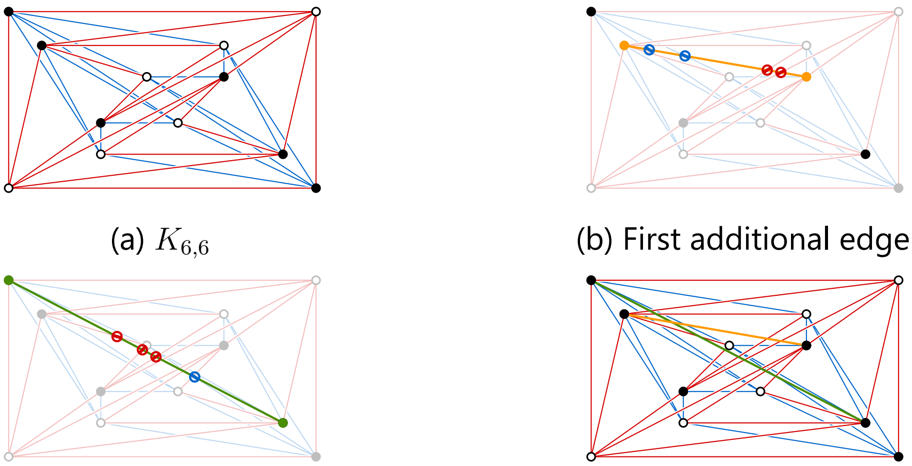
(a) $K_{6,6}$
(b) First additional edge
(c) Second additional edge
(d) Drawing on 4 pages

Figure 1: (a) A Θ^2-drawing of $K_{6,6}$ (reproduced from a figure of Dillencourt, Eppstein and Hirschberg [3]). We can augment this drawing with two additional edges (b-c), which both cross red and blue edges as well as each other. The resulting drawing (d) likely requires four pages.

A very short preliminary example is shown in Figure 1, reproduced from a figure of Dillencourt, Eppstein and Hirschberg [3]. They present a Θ^2-drawing of the complete bipartite graph on 12 vertices $G_{6,6}$. In this drawing we can identify two edges that cross each other as well as at least one edge of each partition. Note that this is not a definitive proof, since one would need to consider all possible partitions of the edges (while the drawing is fixed, the partition is not). Still, it is an indication that (after adding some other edges and one of the two edges of the figure), the resulting drawing will be Θ^3-saturated. In contrast, without a fixed drawing, the complete graph K_{12} has a geometric thickness of only 3 and therefore the number of edges in min- and max-saturated drawings of graphs in Θ^3 would be different.

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Problem 9 Edge Colorings of Complete Geometric Graphs

suggested by Patrick Schnider

The following is a notoriously open problem:

EDGE COLORING COMPLETE GEOMETRIC GRAPHS
Input: A complete geometric graph G on n vertices
Question: How many colors do we need to color the edges of G such that no two crossing edges get the same color.

It is not hard to see that $n-1$ colors always suffice. The best known lower bound is also linear, namely $\frac{n}{2}+1$ [2]. To our embarrassment, we are still unable to prove that cn colors suffice for any $c < 1$. There are however two objects which allow us to improve the upper bound, if the are large enough. The first are non-convex blobs.

Definition 1. *Let P be a set of n points in the plane in general position. The pairwise disjoint subsets $P_1, \ldots, P_4 \subset P$ are called non-convex blobs if for every choice of $p_1 \in P_1, \ldots, p_4 \in P_4$ we have that p_4 lies inside the convex hull of p_1, p_2, p_3.*

It is known that if P contains non-convex blobs, each of size an, then G, the complete geometric graph on P, can be edge-colored with $(1-a)n$ colors [3].

Definition 2. *Let P be a set of n points in the plane in general position. A spoke set of size k on P is a set S of k pairwise non-parallel lines such that in each unbounded region of the arrangement defined by the lines in S there lies at least one point of P.*

It is known that if P contains a spoke set of size bn, then G can be edge-colored with $(1-b)n$ colors [1].

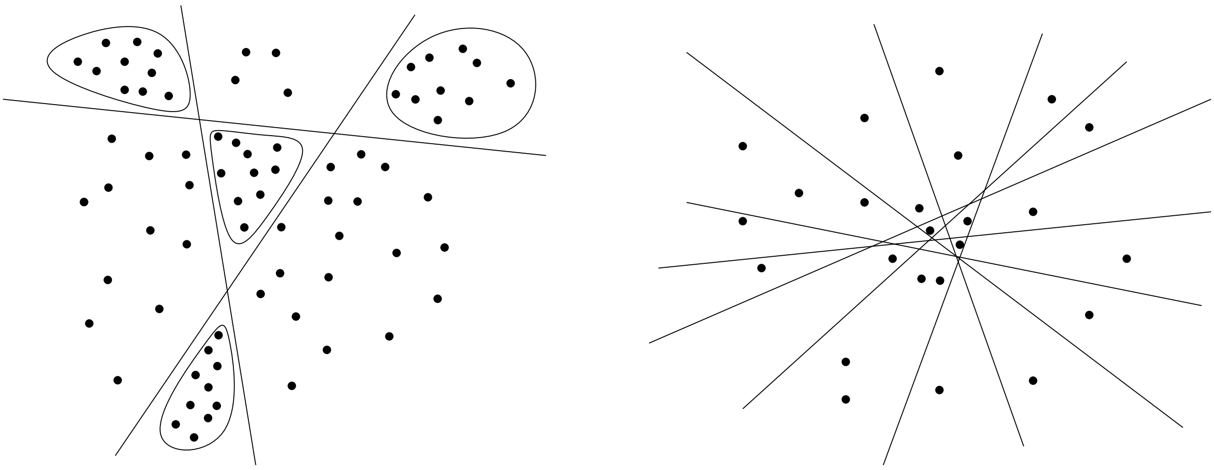

Figure 1: A point set with large non-convex blobs (left) and a point set with a large spoke set (right).

Intuitively, if P has many points that lie “deep”, then we should have large non-convex blobs. On the other hand, if many points are “shallow”, we should be able to find a large spoke set. Can we formalize this?

Question 1. *Let P be a set of n points in the plane in general position and let $b > 0$. Assume that P does not contain non-convex blobs of size bn. Is there an $a > 0$ such that P admits a spoke set of size bn?*

If this question turns out to be too hard, it would also be interesting to find large spoke sets for other definitions of “shallowness” of P, e.g., that all points in P have small Tukey depth (say, $O(\sqrt{n})$).

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Problem 10 Ramsey on Crossing Families

suggested by Joachim Orthaber

Consider a complete geometric graph $\mathcal{D}$ on n points in general position in the plane. A set of k pairwise crossing edges is a *crossing family of size* k in $\mathcal{D}$. It is an open problem whether every complete geometric graph on n vertices contains of crossing family of size linear in n. The best known lower bound is $n^{1-o(1)}$, see Pach, Rubin, and Tardos [2].

As a step towards this problem, let us define a non-crossing family of size k in $\mathcal{D}$ as a collection of 4 disjoint point sets P_1, P_2, P_3, and P_4 of k points in $\mathcal{D}$, each, such that every choice of 4 points $p_i \in P_i$ yields a non-crossing drawing of K_4 with p_4 in the convex hull of the other three points. This notion is inspired by an example of Pach, Saghafian, and Schnider [3].

My question is now the following Ramsey-type relaxation of trying to find a crossing family of linear size:

Question 1. *Does every complete geometric graph on n vertices contain either a crossing family or a non-crossing family of size linear in n?*

This question is inspired by a question from Bose, Hurtado, Rivera-Campo, and Wood [1] on partitioning complete geometric graphs into crossing-free sub-drawings:

Question 2. *Is there a constant $c < 1$ such that every complete geometric graph on n vertices can be partitioned into cn crossing-free sub-drawings?*

It can be shown that a positive answer to Question 1 implies a positive answer to Question 2.

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Problem 11 Reducing to a Triangulation

suggested by Joachim Orthaber

It is well known that every crossing-free sub-drawing of a complete geometric graph on n points in general position can be extended to a *triangulation of the convex hull* of that point set. Other notions of triangulations include the *triangulation of a polygon* or the *triangulation of a pointgon*, that is, a triangulation of a polygon with additional points in its interior, see also Aichholzer, Rote, Speckmann, and Streinu [1].

Instead of extending a crossing-free drawing to a triangulation of its convex hull, my question is now about reducing it to a triangulation of a pointgon (by removing edges).

Question 1. *Which properties are required such that a crossing-free drawing on a point set contains a triangulation of a pointgon on that set?*

In other words, the goal is to remove edges such that all faces of size larger than 3 get connected to a single face and the boundary of that face is still a simple polygon. Clearly the following two properties are necessary to succeed, but are they also sufficient?

- Every vertex is incident to at least one triangular face.
- The union of all triangular faces forms a single polygon with holes.

The above question is actually inspired by a non-geometric setting: In generalized convex drawings every maximal crossing-free sub-drawing fulfills the two stated properties, see Bergold, Felsner, M. Reddy, Orthaber, and Scheucher [2]. That is, they are triangulations (on the sphere) but with some holes in it. Since in the geometric case maximal crossing-free sub-drawing are exactly the triangulations of the convex hull, I wondered whether in this more general case there is also some connection to some kind of triangulations.

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